Math 279 Lecture 17 Notes

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1 Proof of Hairer's Reconstruction Theorem Without Using Wavelets

1.1 Scaling and translation of convolutions

Given a γ-coherent germ $(F_x : x \in \mathbb{R}^d)$, we wish to find a distribution T such that

$$\langle T - F_x, \varphi_x^\delta \rangle \lesssim \delta^\gamma,$$

locally uniformly in x. Recall that coherence means

$$\langle F_x - F_y, \varphi_y^\delta \rangle \lesssim \delta^{-\tau}(|x-y| + \delta)^{\gamma+\tau},$$

locally uniformly in x, y. If it is also uniform in φ with $\|\varphi\|_{C^r} \leq 1$ and $\operatorname{supp} \varphi \subseteq B_1(0)$, then we can find $\beta \geq 0$ such that

$$\langle F_x, \varphi_x^\delta \rangle \lesssim \delta^{-\beta},$$

locally uniformly.

We now give a proof of the existence of T using a single test function φ with $\int \varphi \neq 0$. Here is the strategy for constructing our T. We choose a suitable $\rho \in \mathcal{D}$ with $\int \rho = 1$ and define $\widehat{\rho}_x^n = \rho_x^{2^{-n}} = 2^{dn} \rho(2^n(x-y))$ (recall that $\psi_x^\delta(y) := \delta^{-d}\psi(\frac{y-x}{\delta})$). We will construct ρ that can be represented as $\rho = \psi * \varphi$ for suitable test function ψ and φ that will be determined later. But for now, let us make some observations.

Proposition 1.1.

$$(\psi * \varphi)^\delta = \psi^\delta * \varphi^\delta.$$

Proof.

$$\begin{aligned}
-\delta^{-d}(\psi * \varphi)(\tfrac{x}{\delta}) &= \delta^{-d} \int \psi(\tfrac{x}{\delta} - z)\varphi(z)\, dz \\
&= \delta^{-2d} \int \psi(\tfrac{x}{\delta} - \tfrac{z}{\delta})\varphi(\tfrac{z}{\delta})\, dz \\
&= \int \psi^\delta(x-z)\varphi^d(z)\, dz.
\end{aligned}$$

□

Proposition 1.2.

$$(\psi * \varphi)_x(\cdot) = \int \psi_z(\cdot)\varphi_x(z)\, dz$$

Proof.

$$\begin{aligned}(\psi * \varphi)_x(y) &= \psi * \varphi_x(y)\\ &= \int \psi(y-z)\varphi_x(z)\, dz\\ &= \int \psi_z(y)\varphi_x(z)\, dz.\end{aligned}$$

□

1.2 Construction of T as a limit

In fact, for a carefully selected ρ, we set

$$T_n(x) = F_x(\widehat{\rho}_x^n), \qquad T = \lim_{n\to\infty} T_n,$$

where this limit means $T(\zeta) = \lim_{n\to\infty} \int T_n(x)\zeta(x)\, dx$. For $\gamma > 0$, we show that the limit does exist and satisfies our requirement. For $\gamma < 0$, we need to first get rid of some diverging terms. Again, our ρ takes the form $\rho = \psi * \varphi$ (with ψ and φ to be picked later). To prove our convergence, write

$$T = T_\infty = T_1 + \sum_{n=1}^{\infty}(T_{n+1} - T_n)$$

and show that $|\langle T_{n+1} - T_n, \zeta\rangle| \lesssim 2^{-n\alpha}$ for some $\alpha > 0$. Indeed,

$$\begin{aligned}T_{n+1}(x) - T_n(x) &= F_x(\widehat{\rho}_x^{n+1} - \widehat{\rho}_x^n)\\ &= F_x(\widehat{m}_x^n),\end{aligned}$$

where $m = \rho^{1/2}(y) - \rho(y) = 2^d\rho(2y) - \rho(y)$. Observe that since $\rho = \psi * \varphi$, then

$$m = \rho^{1/2} - \rho = \psi^{1/2} * \varphi^{1/2} - \psi * \varphi.$$

If we chose $\psi = \varphi^2$, then

$$m = \varphi * \varphi^{1/2} - \varphi^2 * \varphi = \varphi * (\varphi^{1/2} - \varphi^2) =: \varphi * \xi.$$

Our goal is bounding $F_x(\widehat{m}_x^n)$. By our propositions,

$$F_x(\widehat{m}_x^n) = F_x\left(\int \widehat{\varphi}_z^n \widehat{\xi}_x^n\, dz\right)$$

$$
\begin{aligned}
&= \int F_x(\widehat{\varphi}^n_z)\widehat{\xi}^n_z(z)\,dz \\
&= A_n + B_n,
\end{aligned}
$$

where

$$
A_n = \int F_z(\widehat{\psi}^n_z)\widehat{\xi}^n_x(z)\,dz, \qquad B_n = \int (F_x - F_z)(\widehat{\psi}^n_z)\widehat{\xi}^n_x(z)\,dz.
$$

Given $\xi \in \mathcal{D}$,

$$
\begin{aligned}
\langle A_n, \zeta\rangle &= \iint F_z(\widehat{\psi}^n_z)\widehat{\xi}^n_x(z)\zeta(x)\,dz\,dx \\
&= \int F_z(\widehat{\varphi}^n_z)(\widehat{\xi}^n + \xi)(z)\,dz.
\end{aligned}
$$

Recall that $\xi = \varphi^{1/2} - \varphi^2$. Imagine that φ satisfies $\int \varphi = 1$, $\int \varphi x^r\,dx = 0$ for $0 < |r| \le \ell$. Hence, $\int \xi x^r\,dx = 0$ for $0 \le |r| \le \ell$. For such φ, we can assert

$$
\widehat{\xi}^n_x + \zeta(z) = \int \widehat{\xi}^n(x-z)\zeta(z)\,dz = \int \widehat{\xi}^n(x-z)\underbrace{(\zeta(z) - P^\ell_x(z))}_{=O(|x-z|^{\ell+1})}\,dz = O(2^{-n(\ell+1)}),
$$

where $P^\ell_x(z)$ is the Taylor expansion up to degree ℓ at x. Thus,

$$
|\langle A_n, \zeta\rangle| \lesssim 2^{n(\beta-\ell-1)},
$$

which is exponentially small if ℓ is sufficiently large. In summary, we have $A = A_\infty = A_1 + \sum_{n=1}^\infty (A_{n+1} - A_n)$ converges as a distribution.

We now turn to the B_ns; this is the one that only converges is $\gamma > 0$. Observe that

$$
|\langle B_n, \zeta\rangle| = \left|\iint (F_x - F_z)(\widehat{\varphi}^n_z)\widehat{\xi}^n_x(z)\,dz\zeta(x)\,dz\right|
$$

Since $|x - z|$ and δ are both of order 2^{-n},

$$
\lesssim 2^{-n\gamma},
$$

which is exponentially small if $\gamma > 0$.

In summary, the limit exists, and we have our candidate for T. It remains to verify that

$$
|\langle T - F_x, \zeta^\delta_x\rangle| \lesssim \delta^\gamma,
$$

locally uniformly. To prove this, observe that since $\rho = \varphi * \psi$, we can write

$$
\begin{aligned}
T(x) &= \lim_{n\to\infty} T_n(x) \\
&= \lim_{n\to\infty} F_x(\widehat{\varphi * \psi}^n)_x
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{n\to\infty} F_x \widehat{\varphi}_y^n \widehat{\psi}_x^n(y)\,dy \\
&= \lim_{n\to\infty} F_x(\widehat{\varphi}_y^n)\psi_x^n(y)\,dy.
\end{aligned}
$$

Also,

$$
\begin{aligned}
F_x(\xi_x^\delta) &= \lim_{n\to\infty} F_x((\xi^\delta * \widehat{\varphi}^n)(x)) \\
&= \lim_{n\to\infty} F_x\left(\int \widehat{\varphi}_y^n \xi_x^\delta(y)\,dy\right) \\
&= \lim_{n\to\infty} \int F_x(\widehat{\varphi}_y^n)\xi_x^\delta(y)\,dy.
\end{aligned}
$$

We will complete the proof next time.